

A UNIVERSAL DIVERGENCE RATE FOR SYMMETRIC BIRKHOFF SUMS IN INFINITE ERGODIC THEORY

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ABSTRACT. We show that there exists a universal gap in the failure of the ergodic theorem for symmetric Birkhoff sums in infinite ergodic theory.

1. INTRODUCTION

For an ergodic infinite measure preserving system, the ergodic theorem fails in the sense that there does not exist a normalizing sequence for its Birkhoff sums [Aar2]. That is for every conservative, ergodic, measure preserving system (X, \mathcal{B}, m, T) with $m(X) = \infty$, $0 \leq f \in L_1(X, m)$ and $a_n \rightarrow \infty$, either

$$\liminf_{n \rightarrow \infty} \frac{S_n(f)}{a_n} = 0 \text{ a.e.}$$

or

$$\limsup_{n \rightarrow \infty} \frac{S_n(f)}{a_n} = \infty \text{ a.e.}$$

Here $S_n(f) := \sum_{k=0}^{n-1} f \circ T^k$ denotes the Birkhoff sum of f . For an invertible transformation one can consider symmetric (two-sided) Birkhoff sums

$$\Sigma_n(f)(x) := \sum_{|k| < n} f(T^k x),$$

where the summation is in a symmetric time interval. The papers [AKW, MS] contain examples of infinite measure preserving transformations for which there exists normalizing constants $a_n \rightarrow \infty$ such that for every $0 \leq f \in L_1(X, m)$,

$$(1.1) \quad \varliminf_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n} > 0 \text{ a.e. and } \varlimsup_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n} < \infty \text{ a.e.}$$

The examples of [AKW] include some natural transformations in infinite ergodic theory such as the class of rank one transformations with bounded cutting sequence and generalized recurrent events with a certain trimmed sum property (some null recurrent Markov chains are in this class). This shows that symmetric Birkhoff sums can behave better than their one sided counterparts. However in the work with Jon Aaronson and Benjamin Weiss we proved that for an invertible infinite measure preserving transformation, there is no ergodic theorem for symmetric Birkhoff sums. That is for every

normalizing sequence $a_n \rightarrow \infty$ and $0 \leq f \in L_1(X, m)$ if $0 < \underline{\lim}_{n \rightarrow \infty} \frac{1}{a_n} \Sigma_n(f)(x) < \infty$, then

$$\underline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n} < \overline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n} \text{ a.e.}$$

The purpose of this note (which is largely taken from the authors Ph.D. thesis) is to prove a universal quantitative divergence rate for symmetric Birkhoff sums.

Theorem 1. *For every conservative, ergodic, measure preserving system (X, \mathcal{B}, m, T) with $m(X) = \infty$, $0 \leq f \in L_1(X, m)$ and $a_n \rightarrow \infty$, if $0 < \underline{\lim}_{n \rightarrow \infty} \frac{1}{a_n} \Sigma_n(f)(x) < \infty$, then*

$$\frac{\underline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}}{\overline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}} \leq \frac{10001}{10002}.$$

After proving the theorem we give an application to the study of fluctuations of symmetric Birkhoff integrals of horocyclic flows on geometrically finite surfaces.

Notation. From now on we will write

$$S_n^-(f) := \sum_{k=1}^{n-1} f \circ T^{-k} = \Sigma_n(f) - S_n(f).$$

For eventually positive sequences a_n, b_n we write:

- $a_n \sim b_n$ if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$.
- $a_n \lesssim b_n$ if $\overline{\lim}_{n \rightarrow \infty} \frac{a_n}{b_n} \leq 1$.
- $a_n \asymp b_n$ if there exists $C > 1$ such that $C^{-1}b_n \leq a_n \leq Cb_n$ for all $n \in \mathbb{N}$.
- For an infinite subset $K \subset \mathbb{N}$, $a_n \underset{n \in K}{\lesssim} b_n$ if $\overline{\lim}_{n \rightarrow \infty, n \in K} \frac{a_n}{b_n} \leq 1$.
- $a = b \pm \epsilon$ means $b - \epsilon < a < b + \epsilon$.
- Given a standard σ -finite measure space (X, \mathcal{B}, m) and a subcollection of sets $\mathcal{C} \subset \mathcal{B}$, we write \mathcal{C}_+ to be the collection of sets $A \in \mathcal{C}$ of positive measure.
- $L_1(X, m)_+$ is the collection of nonnegative integrable functions.
- All transformations or flows in this paper are assumed to be invertible.

2. PRELIMINARIES

Bounded Rational Ergodicity. As in [Aar1], a conservative, ergodic, measure preserving transformation (X, \mathcal{B}, m, T) is called *boundedly rationally ergodic*

(BRE) if $\exists A \in \mathcal{B}$, $0 < m(A) < \infty$ and $M < \infty$ so that

$$(2.1) \quad S_n(1_A)(x) \leq M a_n(A) \text{ a.e. on } A \quad \forall n \geq 1$$

$$\text{where } a_n(A) = \sum_{k=0}^{n-1} \frac{m(A \cap T^{-k}A)}{m(A)^2}.$$

In this case [Aar1], (X, \mathcal{B}, m, T) is *weakly rationally ergodic* (WRE), that is, writing $a_n(T) := a_n(A)$ (where A as in (2.1)), there is a dense hereditary ring

$$\mathcal{R}(T) \subset \mathcal{F} := \{F \in \mathcal{B} : m(F) < \infty\}$$

(including all sets satisfying (2.1)) so that

$$a_n(F) \sim a_n(T) \quad \forall F \in \mathcal{R}(T), \quad m(F) > 0$$

and

$$\sum_{k=0}^{n-1} m(F \cap T^{-k}G) \sim m(F)m(G)a_n(T), \quad \forall F, G \in \mathcal{R}(T).$$

For invertible transformations, one can define similarly the two sided analogs of the properties BRE and WRE. (X, \mathcal{B}, m, T) is:

- two sided, boundedly rationally ergodic if $\exists A \in \mathcal{B}$, $0 < m(A) < \infty$ and $M < \infty$ so that

$$(2.2) \quad \Sigma_n(1_A)(x) \leq M \bar{a}_n(A) \text{ a.e. on } A \quad \forall n \geq 1$$

$$\text{where } \bar{a}_n(A) = \sum_{|k| \leq n} \frac{m(A \cap T^k A)}{m(A)^2} \sim 2a_n(A).$$

- two sided, weakly rationally ergodic if there is a dense hereditary ring

$$\bar{\mathcal{R}}(T) \subset \mathcal{F}$$

(including all sets satisfying (2.2)) so that

$$\bar{a}_n(F) \sim 2a_n(T) \quad \forall F \in \bar{\mathcal{R}}(T), \quad m(F) > 0.$$

If (X, \mathcal{B}, m, T) is one sided BRE then so is $(X, \mathcal{B}, m, T^{-1})$. This can be seen for example by the fact that if $A \in \mathcal{B}$ is the set along which (2.1) holds, then for $n \in \mathbb{N}$ and $x \in A$ one can define

$$k(x, n) := \max \left\{ k \in \mathbb{N} \cup \{0\} : k < n \text{ and } T^{-k}x \in A \right\}.$$

It then follows that for all $n \in \mathbb{N}$ and $x \in A$,

$$\begin{aligned} \sum_{k=-n+1}^0 1_A \circ T^k(x) &= S_{k(x,n)}(1_A) \circ T^{-k(x,n)}(x) \\ &\stackrel{(2.1)}{\leq} M a_{k(x,n)}(A) \quad \text{since } T^{-k(x,n)}x \in A \\ &\leq M a_n(A), \quad \text{since } k(x, n) \leq n. \end{aligned}$$

Therefore in the case of invertible transformations:

- (X, \mathcal{B}, m, T) is two sided BRE if and only if it is (one sided) bounded rationally ergodic.
- If (X, \mathcal{B}, m, T) is two sided BRE, then it is two sided WRE and for $F, G \in \mathcal{R}(T)$,

$$(2.3) \quad \int_F \frac{\Sigma_n(1_G)}{2a_n(T)} dm = \frac{1}{2a_n(T)} \sum_{k=-n}^n m(F \cap T^{-k}G) \\ \sim m(F)m(G) \text{ as } n \rightarrow \infty.$$

2.0.1. *Some observations:* Let (X, \mathcal{B}, m, T) be a conservative, ergodic measure preserving transformation.

- (1) By the ratio ergodic theorem, for all $f, g \in L_1(X, m)$ with $g > 0$,

$$\frac{S_n(f)}{S_n(g)}(x) \xrightarrow{n \rightarrow \infty} \frac{\int_X f dm}{\int_X g dm}, \text{ for a.e. } x$$

and by a similiar argument for T^{-1} ,

$$\frac{\Sigma_n(f)}{\Sigma_n(g)}(x) \xrightarrow{n \rightarrow \infty} \frac{\int_X f dm}{\int_X g dm}, \text{ for a.e. } x.$$

A consequence of this is that in order to check if (1.1) holds for a sequence $a_n \rightarrow \infty$, it is enough to check if it holds for one function $f \in L_1(X, m)_+$. Variants of this application of the ratio ergodic theorem appear throughout this work.

- (2) For $a_n \rightarrow \infty$ and $f \in L_1(X, m)_+$, the functions $\overline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}$ and $\underline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}$ are T invariant, hence constant almost everywhere.
- (3) As in the one sided case (X, \mathcal{B}, m, T) is two sided BRE if and only if for all $f \in L_1(X, m)_+$,

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{2a_n(T)} \Sigma_n(f) < \infty \text{ a.e.}$$

In case T is bounded rationally ergodic, there exists $\underline{\beta} = \underline{\beta}(T) \in [0, 1]$, $\alpha = \alpha(T)$ and $\bar{\beta} = \bar{\beta}(T) \in [1, \infty)$ so that $\forall f \in L_1(X, m)_+$ for m a.e. x :

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \frac{1}{a_n(T)} S_n(f)(x) &= \alpha \int_X f dm \\ \overline{\lim}_{n \rightarrow \infty} \frac{1}{2a_n(T)} \Sigma_n(f)(x) &= \bar{\beta} \int_X f dm \\ \underline{\lim}_{n \rightarrow \infty} \frac{1}{2a_n(T)} \Sigma_n(f)(x) &= \underline{\beta} \int_X f dm. \end{aligned}$$

We will make use of the following proposition from [AKW].

Proposition 2. [AKW, Prop. 1] *Let (X, \mathcal{B}, m, T) be an invertible, conservative, ergodic, measure preserving transformation.*

(i) *If T satisfies (1.1) w.r.t. to some normalizing constants $a_n \rightarrow \infty$, then*

T is bounded rationally ergodic and $a_n \asymp 2a_n(T)$.

(ii) If T is bounded rationally ergodic, then

$$\alpha(T) = \alpha(T^{-1}),$$

whence

$$(2.4) \quad \overline{\beta}(T) \leq \alpha(T) \leq 2\overline{\beta}(T) \text{ and}$$

$$(2.5) \quad \underline{\beta}(T) \leq \frac{\alpha(T)}{2}.$$

A consequence of this proposition is that using the convention that $\frac{a}{\infty} = 0$ for all $0 \leq a < \infty$, if T is not bounded rationally ergodic then for any $a_n \rightarrow \infty$ and $f \in L_1(X, m)_+$ either $\lim_{n \rightarrow \infty} \frac{1}{a_n} \Sigma_n(f)(x) = 0$ or $\overline{\lim}_{n \rightarrow \infty} \frac{1}{2a_n(T)} \Sigma_n(f)(x) = \infty$. Therefore, in order to finish the proof of Theorem 1, we need only consider bounded rationally ergodic transformations.

3. A GAP BETWEEN THE LIMIT INFERIOR AND THE LIMIT SUPERIOR OF SYMMETRIC BIRKHOFF SUMS FOR BRE TRANSFORMATIONS

Theorem 3. *Let (X, \mathcal{B}, m, T) be an infinite, invertible, conservative, ergodic, bounded rationally ergodic, measure preserving transformation, then*

$$\overline{\beta}(T) - \underline{\beta}(T) \geq \frac{1}{5000}.$$

Remark 4. The constant $\delta := \frac{1}{5000}$ was chosen so that

$$(3.1) \quad (1 - 50\delta) \leq 0.99$$

and

$$(3.2) \quad \frac{1}{2} \left(\frac{100}{99} \right)^2 \frac{1 + \delta}{1 - \delta} \leq \frac{1}{\sqrt{3}}.$$

We would like to point out that by a more careful bookkeeping one can obtain a better constant for δ . This will amount in more technical arguments which we chose not to follow. As for now, we don't know of any examples with $\overline{\beta} - \underline{\beta} < \frac{1}{2}$, it is interesting to find out what is the minimal δ so that there exists a conservative, ergodic infinite measure preserving transformation T with $\delta = \overline{\beta}(T) - \underline{\beta}(T)$.

Proof: Suppose otherwise that

$$\overline{\beta}(T) - \underline{\beta}(T) < \delta := \frac{1}{5000}.$$

and let $a(n) := a_n(T)$.

Since $\underline{\beta}(T) \leq 1$ and $\overline{\beta}(T) \geq 1$,

$$1 - \delta < \underline{\beta}(T) \leq 1 \leq \overline{\beta}(T) < 1 + \delta.$$

Consequently for all $A \in \mathcal{F}_+$, a.e. on X ,

$$(3.3) \quad (1 - \delta) m(A) \lesssim \frac{1}{2a_n(T)} \Sigma_n(1_A) \lesssim (1 + \delta) m(A).$$

We claim that

$$(3.4) \quad 2 - 2\delta < \alpha := \alpha(T) < 2 + 2\delta$$

Indeed, by (2.5), $\alpha \geq 2\beta > 2 - 2\delta$ and by (2.4) $\alpha \leq 2\bar{\beta} < 2 + 2\delta$.

The rest of the proof is a quantitative version of the “single orbit” argument in [AKW], which we proceed to specify.

- Fix $A \in \mathcal{F}_+$. By Egorov there exists $B \in \mathcal{F}_+ \cap A$, $m(B) > \frac{3}{4}m(A)$ and $N_0 \in \mathbb{N}$ so that for all $n \geq N_0$ and $x \in B$,

$$(2 - 2\delta)m(A) \leq \sup_{N \geq n} \frac{1}{a(N)} S_N(1_A)(x) \leq (2 + 2\delta)m(A),$$

and

$$(3.5) \quad (2 - 2\delta)a(n)m(A) \leq \Sigma_n(1_A)(x) \leq (2 + 2\delta)a(n)m(A).$$

- Call a point $x \in B$ *admissible* if

$$(A1) \quad \frac{S_n(1_A)(x)}{S_n(1_B)(x)} \xrightarrow{n \rightarrow \infty} \frac{m(A)}{m(B)};$$

$$(A2) \quad \frac{1}{2a(n)} \Sigma_n(1_B)(x) = (1 \pm \delta)m(B), \text{ for all } n \geq N_0;$$

$$(A3) \quad \sup_{N \geq n} \frac{1}{\alpha a(N)} S_N(1_B)(x) \xrightarrow{n \rightarrow \infty} m(B),$$

and there exists $K \subset \mathbb{N}$, an x -*admissible subsequence* in the sense that

$$(A4) \quad T^n x \in B, \forall n \in K \text{ and}$$

$$(A5) \quad \frac{1}{\alpha a(n)} S_n(1_B)(x) \xrightarrow{n \rightarrow \infty, n \in K} m(B).$$

An admissible pair is $(x, K) \in B \times 2^{\mathbb{N}}$ where x is an admissible point and K is an x -admissible subsequence.

Note that if (x, K) is an admissible pair, then by (A1) and (A5)

$$\frac{1}{\alpha a(n)} S_n(1_A)(x) \xrightarrow{n \rightarrow \infty, n \in K} m(A).$$

Lemma 5. *Almost every $x \in B$ is admissible.*

Proof. By (3.5), the definition of α and the ratio ergodic theorem, almost every $x \in B$ satisfies (A1), (A2) and (A3).

Also since $\alpha = \alpha(T) < \infty$, for a.e. $x \in B$, $\exists K \subset \mathbb{N}$ satisfying (A5).

We claim that if $K := \{k_n : n \geq 1\}$, $k_n \uparrow$, then $K' := \{k'_n : n \geq 1\}$ where $k'_n = \max \{j \leq k_n : T^j x \in B\}$ is x -admissible. Evidently K' is infinite and satisfies (A4). To check (A5) for K' :

$$\alpha a(k_n)m(B) \geq \alpha a(k'_n)m(B) \stackrel{(A3)}{\gtrsim} S_{k'_n}(1_B)(x) = S_{k_n}(1_B)(x) \stackrel{(A5)}{\sim} \alpha a(k_n)m(B).$$

This shows that

$$a(k'_n) \sim a(k_n), \text{ as } n \rightarrow \infty$$

and that

$$\frac{1}{\alpha a(k'_n)} S_{k'_n}(1_B)(x) \xrightarrow{n \rightarrow \infty} m(B).$$

□

The proof goes as follows. Lemmas 6, 7 and 8 deal with some consequences of the definition of admissible pairs (x, K) on the growth of the return sequence $a(n)$ along $n \in K$. Then we fix an admissible pair (x, K) and use these Lemmas to arrive to a contradiction.

Lemma 6. *Let $\rho \in (0, 1/2)$. If $x \in B$, $K \subset \mathbb{N}$ and $\{J_n : n \in K\} \subset \mathbb{N}$ satisfy*

$$\begin{aligned} \frac{1}{\alpha a(n)} S_n(1_A)(x) &\xrightarrow{n \rightarrow \infty, n \in K} m(A); \\ n \geq J_n &\xrightarrow{n \rightarrow \infty, n \in K} \infty; \\ \lim_{n \rightarrow \infty, n \in K} \frac{a(J_n)}{a(n)} &\geq \rho, \end{aligned}$$

then

$$\frac{1}{\alpha a(J_n)} S_{J_n}(1_A)(x) \underset{n \in K}{\gtrsim} m(A) \left(1 - \frac{4\delta}{\rho}\right).$$

Proof. Since $x \in B$, for $n \in K$ large

$$S_n(1_A)(x) + S_n^-(1_A)(x) = \Sigma_n(1_A)(x) \lesssim 2(1 + \delta) a(n) m(A).$$

Consequently by (A1) and (A5),

$$S_n(1_A)(x) \sim \alpha a(n) m(A)$$

and

$$\begin{aligned} S_n^-(1_A)(x) &\underset{n \in K}{\lesssim} [2 + 2\delta - \alpha] a(n) m(A) \\ &\leq 4\delta a(n) m(A). \end{aligned}$$

This implies that

$$\begin{aligned} \frac{1}{\alpha a(J_n)} S_{J_n}^-(1_A)(x) &\leq \frac{1}{\alpha a(J_n)} S_n^-(1_A)(x) \underset{n \in K}{\lesssim} \frac{4\delta a(n) m(A)}{\alpha a(J_n)} \\ &\underset{n \in K}{\lesssim} \frac{4\delta}{\alpha \rho} m(A) \end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\alpha a(J_n)} S_{J_n}(1_A)(x) &= \frac{1}{\alpha a(J_n)} \Sigma_{J_n}(1_A)(x) - \frac{1}{\alpha a(J_n)} S_{J_n}^-(1_A)(x) \\
&\stackrel{(A2)}{\gtrsim} \frac{(2-2\delta)}{\alpha} m(A) - \frac{1}{\alpha a(J_n)} S_{J_n}^-(1_A)(x) \\
&\stackrel{\gtrsim}{\underset{n \in K}{\gtrsim}} \frac{1}{\alpha} \left[(2-2\delta) - \frac{4\delta}{\rho} \right] m(A) \geq (1-4\delta/\rho) m(A).
\end{aligned}$$

Here the last inequality follows from

$$\frac{2-2\delta}{\alpha} \geq \frac{2-2\delta}{2+2\delta} \geq 1-2\delta > 1 - \frac{\delta}{\rho},$$

and $\alpha > 2-2\delta > 3/2$. □

Lemma 7. *Let $(x, K) \in B \times 2^{\mathbb{N}}$ be an admissible pair then*

$$\frac{2}{25} \leq \varliminf_{n \rightarrow \infty, n \in K} \frac{a\left(\frac{n}{9}\right)}{a(n)} \ \& \ \varlimsup_{n \rightarrow \infty, n \in K} \frac{a\left(\frac{n}{9}\right)}{a(n)} \leq \frac{1}{3}.$$

Proof. We show first that

$$(a) \quad \varliminf_{n \rightarrow \infty, n \in K} \frac{a\left(\frac{n}{9}\right)}{a(n)} \geq \frac{2}{25}.$$

Define for $n \in K$,

$$J_l := \min \left\{ l \geq \frac{ln}{9} : T^l x \in B \right\} \wedge \frac{(l+1)n}{9}; \quad (0 \leq l \leq 8),$$

then

$$\begin{aligned}
\alpha m(B) a(n) &\stackrel{(A5)}{\lesssim}_{n \in K} S_n(1_B)(x) = \sum_{l=0}^8 S_{\frac{n}{9}}(1_B) \left(T^{\frac{ln}{9}} x \right) \\
&= \sum_{l=0}^8 S_{\frac{(l+1)n}{9} - J_l}(1_B) (T^{J_l} x) \leq \sum_{l=0}^8 S_{\frac{n}{9}}(1_B) (T^{J_l} x) \\
&\leq \sum_{l=0}^8 S_{\frac{n}{9}}(1_B) (T^{J_l} x) \leq \sum_{l=0}^8 \left\| S_{\frac{n}{9}}(1_A) \right\|_{L_\infty(B)} \\
&\lesssim 9(2+2\delta) a\left(\frac{n}{9}\right) m(A).
\end{aligned}$$

Thus

$$\varliminf_{n \rightarrow \infty, n \in K} \frac{a\left(\frac{n}{9}\right)}{a(n)} \geq \frac{\alpha m(B)}{18(1+\delta)m(A)} > \frac{2}{25}. \quad \square (a)$$

Next we show

$$(b) \quad \varlimsup_{n \rightarrow \infty, n \in K} \frac{a\left(\frac{n}{9}\right)}{a(n)} \leq \frac{1}{\sqrt{3}}.$$

By (a) and monotonicity of $a(n)$, $\{J_n = n/3 : n \in K\}$ satisfies the conditions of Lemma 6 with $\rho = 2/25$, hence

$$S_{\frac{n}{3}}(1_A)(x) \underset{n \in K}{\gtrsim} \alpha a\left(\frac{n}{3}\right) (1 - 50\delta) m(A).$$

By (A1),

$$(3.6) \quad S_{\frac{n}{3}}(1_B)(x) \underset{n \in K}{\gtrsim} \alpha a\left(\frac{n}{3}\right) (1 - 50\delta) m(B) \stackrel{(3.1)}{\geq} \frac{99\alpha}{100} a\left(\frac{n}{3}\right) m(B).$$

For $n \in K$, let

$$j_n := \max \{j \leq n/3 : T^j x \in B\}.$$

We claim that $a(j_n) \underset{n \in K}{\gtrsim} 0.99a(n/3)$, since

$$\begin{aligned} \alpha a(j_n) m(B) &\gtrsim S_{j_n}(1_B)(x) = S_{\frac{n}{3}}(1_B)(x) \\ &\underset{n \in K}{\gtrsim} \frac{99\alpha}{100} a\left(\frac{n}{3}\right) m(B). \end{aligned}$$

Finally since $T^{j_n} x \in B$,

$$\begin{aligned} (2 + 2\delta) a(n) m(A) &\gtrsim \Sigma_n(1_A)(T^{j_n} x) = \sum_{k=-n+j_n}^{n+j_n} 1_A(T^k x) \\ &\geq \Sigma_{j_n}(1_A)(T^{j_n} x) + \Sigma_{j_n}(1_A)(T^n x) \\ &\stackrel{(\star)}{\gtrsim} 2(2 - 2\delta) a(j_n) m(A) \underset{n \in K}{\gtrsim} (4 - 4\delta) \left(0.99a\left(\frac{n}{3}\right)\right) m(A). \end{aligned}$$

In (\star) we used the fact that $T^{j_n} x, T^n x \in B$. Therefore

$$\overline{\lim}_{n \rightarrow \infty, n \in K} \frac{a(n/3)}{a(n)} \leq \frac{100}{198} \cdot \frac{(1 + \delta)}{(1 - \delta)} \stackrel{(3.2)}{\leq} \frac{1}{\sqrt{3}}, \quad \square (b)$$

Next, we show that

$$(c) \quad \overline{\lim}_{n \rightarrow \infty, n \in K} \frac{a\left(\frac{n}{9}\right)}{a(n)} \leq \frac{1}{3}.$$

For $n \in K$, let

$$L_n := \min \left\{ J \geq \frac{n}{3} : T^J x \in B \right\}.$$

Since $n \in K$, $T^n x \in B$, whence $L_n \leq n$.

It follows from

$$\frac{a(L_n)}{a(n)} \geq \frac{a(n/9)}{a(n)} \underset{n \in K}{\gtrsim} \frac{2}{25},$$

and Lemma 6 that (here we move from 1_A to 1_B using condition (A1)),

$$\begin{aligned} \alpha (1 - 50\delta) m(B) a(L_n) &\underset{n \in K}{\lesssim} S_{L_n}(1_B)(x) \leq S_{n/3}(1_B)(x) + 1 \\ &\lesssim \alpha a(n/3) m(B), \end{aligned}$$

hence

$$(3.7) \quad a(n/3) \underset{n \in K}{\gtrsim} 0.99a(L_n).$$

Define for $n \in K$,

$$l_n := \max \left\{ j \leq \frac{L_n}{3} : T^j x \in B \right\}.$$

By repeating the previous argument with L_n replaced by $L_n/3$ (which is still greater or equal to $n/9$), by Lemma 6,

$$S_{L_n/3}(1_A)(x) \underset{n \in K}{\gtrsim} \frac{99\alpha}{100} a(L_n/3) \cdot m(A)$$

and

$$\begin{aligned} \alpha a(l_n) m(B) &\underset{n \in K}{\gtrsim} S_{l_n}(1_B)(x) = S_{L_n/3}(1_B)(x) \\ &\underset{n \in K}{\gtrsim} \alpha \cdot (0.99) a(L_n/3) m(B). \end{aligned}$$

Therefore

$$(3.8) \quad a(l_n) \underset{n \in K}{\gtrsim} 0.99a(L_n/3).$$

The argument in the proof of (b) shows that

$$\begin{aligned} 2(2-2\delta) a(l_n) m(A) &\lesssim \Sigma_{l_n}(1_A)(T^{l_n}x) + \Sigma_{l_n}(1_A)(T^{L_n}x) \\ &\leq \Sigma_{L_n}(1_A)(T^{l_n}x) \lesssim (2+2\delta) a(L_n) m(A). \end{aligned}$$

Here we used in the first inequality the fact that $T^{l_n}x, T^{L_n}x \in B$ and in the last inequality the fact $T^{l_n}x \in B$.

Therefore

$$(3.9) \quad a(l_n) \underset{n \in K}{\lesssim} \left(\frac{1}{2}\right) \left(\frac{1+\delta}{1-\delta}\right) a(L_n),$$

and

$$\begin{aligned} \frac{a(n/9)}{a(n/3)} &\leq \frac{a(L_n/3)}{a(n/3)} \stackrel{(3.7)}{\leq} \frac{100}{99} \cdot \frac{a(L_n/3)}{a(L_n)} \\ &\stackrel{(3.8)}{\lesssim} \left(\frac{100}{99}\right)^2 \frac{a(l_n)}{a(L_n)} \stackrel{(3.9)}{\lesssim} \left(\frac{100}{99}\right)^2 \cdot \frac{1}{2} \left(\frac{1+\delta}{1-\delta}\right) \\ &\stackrel{(3.2)}{\leq} \frac{1}{\sqrt{3}}. \end{aligned}$$

Finally

$$\frac{a(n/9)}{a(n)} = \frac{a(n/9)}{a(n/3)} \cdot \frac{a(n/3)}{a(n)} \underset{n \in K}{\lesssim} \frac{1}{3}. \quad \square(c)$$

□

Lemma 8. *If (x, K) is an admissible pair then*

$$\overline{\lim}_{n \rightarrow \infty, n \in K} \frac{a\left(\frac{8n}{9}\right)}{a(n)} \leq 0.94.$$

Proof. First we show that

$$(3.10) \quad S_{\frac{n}{9}}^-(1_A)(T^n x) \underset{K \ni n \rightarrow \infty}{\gtrsim} (2 - 52\delta) a\left(\frac{n}{9}\right) m(A) \geq \frac{96\alpha}{100} a\left(\frac{n}{9}\right) m(A),$$

here the last inequality follows from $\alpha \leq 2 + 2\delta = \frac{10002}{5000}$ and $(2 - 52\delta) = \frac{9648}{5000} \geq \frac{96\alpha}{100}$.

Indeed, since

$$S_n^-(1_A)(T^n x) = S_n(1_A)(x) \underset{K \ni n \rightarrow \infty}{\sim} \alpha a(n) m(A),$$

then

$$\begin{aligned} S_n(1_A)(T^n x) &= \Sigma_n(1_A)(T^n x) - S_n^-(1_A)(T^n x) \\ &\underset{K \ni n \rightarrow \infty}{\sim} \Sigma_n(1_A)(T^n x) - \alpha a(n) m(A). \end{aligned}$$

In addition for every $n \in K$, $T^n x \in B$, it follows from (A2) that as $K \ni n \rightarrow \infty$,

$$\Sigma_n(1_A)(T^n x) \lesssim (2 + 2\delta) a(n) m(A).$$

Therefore since $\alpha > 2 - 2\delta$,

$$(3.11) \quad S_n(1_A)(T^n x) \underset{K \ni n \rightarrow \infty}{\lesssim} ((2 + 2\delta) - \alpha) a(n) m(A) \leq 4\delta a(n) m(A).$$

Finally

$$\begin{aligned} S_{\frac{n}{9}}^-(1_A)(T^n x) &\geq \Sigma_{\frac{n}{9}}(1_A)(T^n x) - S_n^-(1_A)(T^n x) \\ &\stackrel{\text{"(A2) and (3.11)"}}{\gtrsim}_{n \in K} \left[(2 - 2\delta) a\left(\frac{n}{9}\right) - 4\delta a(n) \right] m(A) \\ &\stackrel{\text{Lemma 7}}{\gtrsim}_{n \in K} \left[(2 - 2\delta) a\left(\frac{n}{9}\right) - 50\delta a\left(\frac{n}{9}\right) \right] m(A) \\ &= (2 - 52\delta) m(A) a\left(\frac{n}{9}\right). \quad \square(3.10) \end{aligned}$$

Now since

$$\frac{a\left(\frac{8n}{9}\right)}{a(n)} \geq \frac{a\left(\frac{n}{9}\right)}{a(n)} \underset{n \in K}{\gtrsim} \frac{2}{25},$$

then by Lemma 6,

$$\begin{aligned}
\alpha(1-50\delta)a\left(\frac{8n}{9}\right)m(A) &\underset{n \in K}{\lesssim} S_{\frac{8n}{9}}(1_A)(x) = S_n(1_A)(x) - S_{\frac{n}{9}}^-(1_A)(T^n x) \\
&\stackrel{(3.10) \text{ and } (A5)}{\underset{n \in K}{\lesssim}} \alpha m(A) \left[a(n) - \frac{96}{100} a\left(\frac{n}{9}\right) \right] \\
&\stackrel{\text{Lemma 7}}{\underset{n \in K}{\lesssim}} \alpha m(A) a(n) \left[1 - \frac{96}{100} \cdot \frac{2}{25} \right] = \alpha m(A) a(n) \left[\frac{93}{100} \right].
\end{aligned}$$

Whence

$$\frac{a\left(\frac{8n}{9}\right)}{a(n)} \underset{n \in K}{\lesssim} \frac{93}{100(1-50\delta)} \leq 0.94.$$

□

Proof of Theorem 3. Fix an admissible pair $(x, K) \in B \times 2^{\mathbb{N}}$, then by Lemmas 7 and 8,

$$\overline{\lim}_{n \rightarrow \infty, n \in K} \frac{a\left(\frac{n}{9}\right)}{a(n)} \leq \frac{1}{3} \quad \text{and} \quad \overline{\lim}_{n \rightarrow \infty, n \in K} \frac{a\left(\frac{8n}{9}\right)}{a(n)} \leq 0.94.$$

For $n \in K$, let

$$\mathfrak{J}_n = \mathfrak{J}_n(x) := \min \left\{ j \geq \frac{n}{9} : T^j x \in B \right\}.$$

We claim that $\mathfrak{J}_n \leq \frac{8n}{9}$; else $\frac{8n}{9} < \mathfrak{J}_n \leq n$ (since for $n \in K$, $T^n x \in B$) and therefore as $n \rightarrow \infty$, $n \in K$:

$$\begin{aligned}
\alpha a(n) m(B) &\sim S_n(1_B)(x) \\
&= S_{\mathfrak{J}_n}(1_B)(x) + S_{n-\mathfrak{J}_n \vee 0}(1_B)(T^{\mathfrak{J}_n} x) \\
&\leq S_{\frac{n}{9}}(1_B)(x) + 1 + S_{n-\mathfrak{J}_n \vee 0}(1_B)(T^{\mathfrak{J}_n} x) \\
&\stackrel{(\diamond)}{\leq} S_{\frac{n}{9}}(1_B)(x) + 1 + S_{\frac{n}{9}}(1_B)(T^{\mathfrak{J}_n} x) \\
&\stackrel{(A3)}{\lesssim} 2\alpha a\left(\frac{n}{9}\right) m(B).
\end{aligned}$$

The inequality of (\diamond) is where we assume in the contranegative that $\mathfrak{J}_n \geq \frac{8n}{9}$.

Thus

$$\frac{1}{2} \underset{n \in K}{\lesssim} \frac{a\left(\frac{n}{9}\right)}{a(n)} \underset{n \in K}{\lesssim} \frac{1}{3}.$$

This contradiction shows that $\mathfrak{J}_n \leq \frac{8n}{9}$.

Finally since for large $n \in K$, $\frac{n}{9} \leq \mathfrak{J}_n \leq \frac{8n}{9}$:

$$[0, n] \subset \left[\mathfrak{J}_n - \frac{8n}{9}, \mathfrak{J}_n + \frac{8n}{9} \right].$$

Therefore as $n \rightarrow \infty$, $n \in K$,

$$\begin{aligned} (2 + 2\delta) a \left(\frac{8n}{9} \right) m(A) &\stackrel{T^{\mathfrak{I}n}(x) \in B}{\gtrsim} \Sigma_{\frac{8n}{9}}(1_A) \left(T^{\mathfrak{I}n} x \right) \geq S_n(1_A)(x) \\ &\underset{n \in K}{\sim} \alpha a(n) m(A) \geq (2 - 2\delta) a(n) m(A), \end{aligned}$$

whence by Lemma (8),

$$\frac{1 - \delta}{1 + \delta} \underset{n \in K}{\lesssim} \frac{a \left(\frac{8n}{9} \right)}{a(n)} \underset{n \in K}{\lesssim} 0.94.$$

This is a contradiction since $\frac{1 - \delta}{1 + \delta} = \frac{4999}{5001} > 0.94$. This proves the theorem. \square

4. THE MAIN STEP TO MOVE FROM A RETURN SEQUENCE TO A UNIVERSAL BOUND

Lemma 9. *Let (X, \mathcal{B}, m, T) be an infinite, invertible, conservative, bounded rationally ergodic, measure preserving transformation then for any sequence $a_n \rightarrow \infty$ and for all $f \in L_1(X, m)_+$ if $0 < \underline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n} < \infty$, then*

$$\frac{\underline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}}{\overline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}} \leq \sqrt{\frac{\beta(T)}{\underline{\beta}(T)}}.$$

Proof. Let $a_n \rightarrow \infty$. Assume in the contranegative that for one (equivalently for all) $0 \leq f \in L^1(X, m)_+$,

$$\frac{\underline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}}{\overline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}} > \sqrt{\frac{\beta(T)}{\underline{\beta}(T)}}$$

$\overline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n} > 0$ and $\underline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n} < \infty$. Notice that this means that for all $f \in L_1(X, m)$, $\overline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n} < \infty$.

Since (X, \mathcal{B}, m, T) is bounded rationally ergodic, there exists $A \in \bar{\mathcal{R}}(T)$ with

$$0 < m(A) < \infty.$$

By multiplying a_n by constants we can assume that,

$$\overline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(1_A)}{a_n} = m(A) \text{ and } \underline{\lim}_{n \rightarrow \infty} \frac{\Sigma_n(1_A)}{a_n} > \sqrt{\frac{\beta(T)}{\underline{\beta}(T)}} m(A).$$

As before, it follows from Egorov's theorem that for all $\gamma < \sqrt{\frac{\beta(T)}{\underline{\beta}(T)}} < 1 < \lambda$, there exists $B \subset A$ of positive measure so that for all n large,

$$\gamma m(A) \leq \frac{\Sigma_n(1_A)(x)}{a_n} \leq \lambda m(A) \text{ uniformly in } x \in B,$$

and thus for large n

$$\gamma m(A) m(B) \leq \int_B \frac{\Sigma_n(1_A)(x)}{a_n} dm \leq \lambda m(A) m(B).$$

Since $\bar{\mathcal{R}}(T)$ is hereditary, $B \in \bar{\mathcal{R}}(T)$. It follows from (2.3) that,

$$\begin{aligned} \int_B \frac{\Sigma_n(1_A)(x)}{a_n} dm &= \frac{2a_n(T)}{a_n} \int_B \frac{\Sigma_n(1_A)(x)}{2a_n(T)} dm \\ &\sim \frac{2a_n(T)}{a_n} m(A)m(B). \end{aligned}$$

This shows that

$$\gamma a_n \lesssim 2a_n(T) \lesssim \lambda a_n.$$

Consequently for all $0 \leq f \in L^1(X, m)$,

$$\begin{aligned} \frac{\lim_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}}{\lim_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}} &\leq \frac{\lambda \lim_{n \rightarrow \infty} \frac{\Sigma_n(f)}{2a_n(T)}}{\gamma \lim_{n \rightarrow \infty} \frac{\Sigma_n(f)}{2a_n(T)}} \\ &= \frac{\lambda \beta(T)}{\gamma \beta(T)} \end{aligned}$$

Since γ is arbitrary close to $\sqrt{\frac{\beta(T)}{\beta(T)}}$ and λ is arbitrarily close to 1,

$$\sqrt{\frac{\beta(T)}{\beta(T)}} < \frac{\lim_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}}{\lim_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}} \leq \sqrt{\frac{\beta(T)}{\beta(T)}}, \forall f \in L_1(X, m)_+$$

a contradiction. \square

Remark 10. In [AKW] we considered two important subclasses of infinite measure preserving transformations. Namely the “Rank one transformations” and “transformations admitting a generalized recurrent event” (the latter includes the class of null recurrent Markov shifts). In those examples when (1.1) happens then

$$\frac{\beta(T)}{\beta(T)} \leq \frac{1}{2}.$$

This together with the previous Lemma shows that for those examples for all $a_n \rightarrow \infty$ and $f \in L_1(X, m)_+$, if $0 < \lim_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n} < \infty$, then

$$\frac{\lim_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}}{\lim_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}} \leq \frac{1}{\sqrt{2}}.$$

5. PROOF OF THEOREM 1

Let (X, \mathcal{B}, m, T) be a conservative, ergodic, measure preserving transformation with $m(X) = \infty$ and $a_n \rightarrow \infty$ such that for all $f \in L_1(X, m)_+$

$$0 < \lim_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n} < \infty.$$

It follows from the comment after Proposition 2 that T is bounded rationally ergodic. By Lemma 9,

$$\frac{\lim_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}}{\lim_{n \rightarrow \infty} \frac{\Sigma_n(f)}{a_n}} \leq \sqrt{\frac{\beta(T)}{\bar{\beta}(T)}}$$

and by Theorem 3 one has

$$\bar{\beta}(T) - \underline{\beta}(T) \geq \frac{1}{5000}.$$

The theorem follows from

$$\begin{aligned} \frac{\beta(T)}{\bar{\beta}(T)} &\leq \sup \left\{ \frac{y}{x} : y < 1 < x, |x - y| > \frac{1}{5000} \right\} \\ &= \sup \left\{ \frac{y}{y + 1/5000} : y < 1 \right\} \\ &= \frac{5000}{5001} \end{aligned}$$

and

$$\sqrt{\frac{5000}{5001}} \leq 1 - \frac{1}{10002}.$$

6. APPLICATIONS FOR HOROCYCLIC FLOWS ON GEOMETRICALLY FINITE HYPERBOLIC SPACES

In [MS], Maucourant and Schapira considered the horocycle flow on geometrically finite hyperbolic spaces and showed examples where the invariant measure is infinite yet one still has precise knowledge of the fluctuations of the symmetric Birkhoff integrals which we now proceed to specify.

In this setting, let Γ_0 be a non elementary finitely generated discrete subgroup of $G = SL(2, \mathbb{R})$ without Torsion elements other than $-Id$. Equivalently the surface $S = \Gamma_0 \backslash \mathbb{H}$ where \mathbb{H} is the hyperbolic plane, is a geometrically finite hyperbolic surface. On the tangent bundle of S one can consider two measures. The first is the measure of maximal entropy for the geodesic flow, also called the Bowen-Margulis or Patterson Sullivan measure which we will denote by m^{ps} . This measure is supported on Ω , the non wandering set of the geodesic flow. The non wandering set \mathcal{E} of the horocyclic flow is the union of horocycles intersecting Ω . By [Bu, Ro], the horocyclic flow has a unique ergodic invariant probability measure of full support on \mathcal{E} . This measure, denoted by m , is often called the Burger-Roblin measure. The critical exponent of $\Gamma := \pi_1(S)$ is defined by

$$\delta := \limsup_{T \rightarrow \infty} \log \frac{1}{T} \# \{ \gamma \in \Gamma_0 : d(o, \gamma o) \leq T \},$$

for any fixed point $o \in \mathbb{H}$. In words δ is the exponential growth rate of the orbits of Γ on \mathbb{H} . The ergodic theorem of [MS] is the following (We took the liberty of rephrasing it in a way that will explain the connection with symmetric Birkhoff sums).

Theorem. [MS](1) *Let S be a non elementary geometrically finite hyperbolic surface. Let $u \in \mathcal{E}$ be a non periodic and non wandering vector for the horocyclic flow. If $f : T^1S \rightarrow \mathbb{R}$ is continuous with compact support, then*

$$\lim_{t \rightarrow \infty} \frac{1}{m_{H^-(u)} \left((h^s u)_{|s| \leq t} \right)} \int_{-t}^t f(h^s u) ds = \frac{1}{m^{ps}(T^1S)} \int_{T^1S} f dm.$$

Here $m_{H^-(u)}$ is the conditional measure of the Patterson-Sullivan measure on the strong stable horocycle $H^-(u) = (h^s u)_{s \in \mathbb{R}}$.

(2) Writing $\tau(u) := m_{H^-(u)} \left((h^s u)_{|s| \leq 1} \right)$, then τ is continuous and $m_{H^-(u)} \left((h^s u)_{|s| \leq t} \right) = t^\delta \tau(g^{\log t} u)$.

(3) If S is convex cocompact, the non wandering set $\Omega \subset \mathcal{E}$ of the geodesic flow is compact, the map τ is bounded from above and below on Ω . Thus there exists constants $c_S, C_S > 0$ such that

$$\frac{c_S t^\delta}{m^{ps}(T^1S)} \int_{T^1S} f dm \lesssim \int_{-t}^t f(h^s u) ds \lesssim \frac{C_S t^\delta}{m^{ps}(T^1S)} \int_{T^1S} f dm, \text{ as } t \rightarrow \infty$$

The question arises of how close to 1 can $\frac{c_S}{C_S}$ be? For example is it true that there exists a sequence of convex cocompact geometrically finite surfaces $S_n = \Gamma_n \backslash \mathbb{H}$ such that

$$\frac{c_{S_n}}{C_{S_n}} \xrightarrow{n \rightarrow \infty} 1?$$

By modifying our proof for flows one sees that the answer to the last question is negative. The proof carries on verbatim once one makes the following adjustments:

- Definition of bounded rational ergodicity for flows by saying that a measure preserving flow $(X, \mathcal{B}, m, \{\phi_s\}_{s \in \mathbb{R}})$ is bounded rationally ergodic if there exists a set $A \in \mathcal{B}$, $0 < m(A) < \infty$ with $M > 0$ such that for all $x \in A$ and $T > 0$,

$$\int_0^T 1_A \circ \phi_s(x) ds \leq M a_T(A)$$

where $a_T(A) := \frac{1}{m(A)^2} \int_0^T m(A \cap \phi_{-s} A) ds$

- Showing that if for a monotone increasing function $a : [0, \infty) \rightarrow [0, \infty)$ and a set $A \subset \mathcal{E}$ of positive m - measure,

$$(6.1) \quad 0 < c \lesssim \frac{1}{a(t)} \int_0^t 1_A(h^s(u)) dt \lesssim C < \infty$$

for m a.e. $u \in \mathcal{E}$, then the functions

$$\mathcal{E} \times [0, \infty) \ni (u, t) \mapsto F_t(u) := \frac{1}{a(t)} \int_0^t 1_A(h^s(u)) dt$$

satisfy the conditions of the Egorov type theorem for continuous parameter flows. In fact this case is much simpler and can be verified by applying Egorov on a discretization of the time parameter (a

discrete skeleton) and then using the equicontinuity in t of the map F_t .

- By the previous step one can carry the proof verbatim by first showing that the flow is bounded rationally ergodic and then applying our argument on a single orbit with minor modifications (in the definition of the stopping times).

The concluding statement is as follows.

Corollary 11. *There exists a universal $\epsilon > 0$ such that for any S a convex cocompact geometrically finite hyperbolic surface*

$$\frac{c_S}{C_S} > 1 - \epsilon$$

where c_S, C_S are the constants defined by

$$c_S := \frac{\liminf_{t \rightarrow \infty} \frac{1}{t^{\delta(S)}} \int_{-t}^t f(h^s u) ds}{\frac{1}{m^{ps}(T^1 S)} \int_{T^1 S} f dm} \quad m - a.e. \quad u \in \mathcal{E}$$

and

$$C_S := \frac{\limsup_{t \rightarrow \infty} \frac{1}{t^{\delta(S)}} \int_{-t}^t f(h^s u) ds}{\frac{1}{m^{ps}(T^1 S)} \int_{T^1 S} f dm} \quad m - a.e. \quad u \in \mathcal{E},$$

for any $f : T^1 S \rightarrow \mathbb{R}$ continuous with compact support. Equivalently

$$\begin{aligned} c_S &:= \operatorname{ess-lim\,inf}_{T \rightarrow \infty} \tau \left(g^{\log T} u \right) \\ C_S &:= \operatorname{ess-lim\,sup}_{T \rightarrow \infty} \tau \left(g^{\log T} u \right). \end{aligned}$$

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